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DAMAGE-PLASTICITY IN A UNIAXIALLY LOADED COMPOSITE LAMINA: OVERALL ANALYSIS

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Abstract—The initiation and evolution of damage and elasto-plastic deformation in composite laminae is studied using an overall approach. In this approach, one damage tensor is used to describe damage in composite materials including the initiation, growth, and coalescence of voids and cracks in the matrix, fiber fracture, and debonding. The mathematical formulation is derived using vectors and matrices to keep the mathematics accessible to many readers who may not be familiar with tensor analysis. The model is applied to a thin fiber-reinforced unidirectional composite lamina subjected to uniaxial tension. This problem is selected in order to demonstrate how the model can be used without resorting to the finite element method. A governing system of nine simultaneous ordinary differential equations is established for this problem based on the overall approach to damage in composite materials. The system is solved numerically and the results are discussed.

INTRODUCTION

The analysis and modeling of damage mechanisms in composite materials are now the subjects of extensive research. However, most of the available material is limited in scope where no distinct correlation can be made among the various methods. In particular, there is no consistent and systematic approach to model damage in composite materials.

Most of the available damage models fall into two major categories. The first category includes all models using a continuum approach (Talreja, 1985, 1986; Christensen, 1990; Arnold and Kruch, 1991). The second category consists of all other models that use some sort of micromechanical characterization (Dvorak and Bahei-El-Din, 1979, 1982, 1987). Allen and Harris (1987) and Allen *et al.* (1987) analysed distributed damage in elastic composites with thermal effects. However, they did not consider the plasticity of the matrix material. Other works are also available (Dvorak *et al.*, 1985; Dvorak and Laws, 1987; Laws and Dvorak, 1987; Allen *et al.*, 1988; Boyd *et al.*, 1993) that deal with this problem either partially or inconclusively. The recent work of Voyiadjis and Kattan (1993b) attempts to present a consistent formulation of a damage theory for fiber-reinforced composite materials.

Kachanov (1958) proposed a damage theory for continuous media that started what is now known as damage mechanics. After Kachanov's pioneering work, Chaboche (1988a, 1988b), Lemaitre (1984, 1986), Sidoroff (1981), Lee *et al.* (1985), and Murakami (1988) applied the theory to various types of damage mechanisms. Recently, Lemaitre (1985), Kattan and Voyiadjis (1990, 1993a), and Voyiadjis and Kattan (1990) extended the theory to the coupling of damage and plastic deformation. The application of damage mechanics to composite materials is also being investigated; Kattan and Voyiadjis (1993b) applied it to the analysis of elastic damage in uniaxially loaded unidirectional fiber-reinforced composite laminae.

In this study, the recent work of Kattan and Voyiadjis (1993c) is applied to the problem of damage initiation and growth in a uniaxially loaded unidirectional fiber-reinforced composite lamina. Damage is modeled according to the overall approach in which one damage matrix is used to describe the various damage mechanisms in the composite system. A governing system of nine simultaneous ordinary differential equations is established for this problem. The system is solved numerically and the results are discussed. The formulation of the model is cast using vectors and matrices without the use of tensor analysis. This problem is selected because it can be solved numerically without the use of the finite element method. A subsequent paper will demonstrate the implementation of the model using finite elements.

THEORETICAL FORMULATION

An overall approach to the characterization of damage in elasto-plastic fiber-reinforced metal matrix composites (Kattan and Voyiadjis, 1993c) is formulated using simple mathematical techniques. The model is cast in a simple form without use of tensors or advanced mathematics. In the formulation, the brackets [ ] are used to denote 3×3 matrices, while the braces { } are used to denote 3×1 vectors. The superscript T indicates the transpose of a vector or matrix. The formulation is general, except that the only restriction is the formula for the derivative of the damage effect matrix [M] which is valid only for problems involving principal damage variables (e.g. uniaxial tension). The composite system consists of an elastoplastic matrix reinforced by continuous, perfectly aligned cylindrical fibers.

Let $\{\bar{\sigma}^m\}$ and $\{\bar{\sigma}^f\}$ be the matrix and fiber effective stress vectors, respectively. In the formulation given here, $\{\bar{\sigma}^m\}$ and $\{\bar{\sigma}^f\}$ take the form

$$\{\bar{\sigma}^m\} = [\bar{\sigma}_1^m \bar{\sigma}_2^m \bar{\sigma}_3^m]^T \tag{1a}$$

$$\{\bar{\sigma}^f\} = [\bar{\sigma}_1^f \bar{\sigma}_2^f \bar{\sigma}_3^f]^T. \tag{1b}$$

Similarly, the overall effective stress vector $\{\bar{\sigma}\}$ takes the form

$$\{\bar{\sigma}\} = [\bar{\sigma}_1 \bar{\sigma}_2 \bar{\sigma}_3]^T. \tag{1c}$$

The elastic stress concentration matrices $[\bar{B}^m]$ and $[\bar{B}^f]$ for the matrix and fibers are defined as follows:

$$\{\bar{\sigma}^m\} = [\bar{B}^m]\{\bar{\sigma}\} \tag{2a}$$

$$\{\bar{\sigma}^f\} = [\bar{B}^f]\{\bar{\sigma}\}, \tag{2b}$$

where $[\bar{B}^m]$ and $[\bar{B}^f]$ are 3×3 constant matrices. For the case of plastic loading or elastic unloading, eqn (2a) is rewritten in incremental form as follows:

$$\{d\bar{\sigma}^m\} = [\bar{B}^{mp}]\{d\bar{\sigma}\}, \tag{2c}$$

where $[\bar{B}^{mp}]$ is a 3×3 elasto-plastic stress concentration matrix for the matrix material. Several models are available in the literature for the determination of the three matrices $[\bar{B}^m]$, $[\bar{B}^f]$ and $[\bar{B}^{mp}]$ (Dvorak and Bahei-El-Din, 1979, 1982, 1987).

Let $\{\bar{\tau}\}$ and $\{\bar{\tau}^m\}$ be the overall and matrix deviatoric vectors, respectively. Then they are related to the total overall and matrix vectors $\{\bar{\sigma}\}$ and $\{\bar{\sigma}^m\}$ as follows:

$$\{\bar{\tau}\} = [a]\{\bar{\sigma}\} \tag{3a}$$

$$\{\bar{\tau}^m\} = [a]\{\bar{\sigma}^m\}, \tag{3b}$$

where the constant 3×3 matrix $[a]$ is given by

$$[a] = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}. \tag{3c}$$

The matrix $[a]$ is idempotent, i.e. $[a]^n = [a]$, where n is a positive integer. In particular, the relation $[a]^2 = [a]\,[a] = [a]$ will be used in the present formulation.

Effective stresses and the yield function

The relations governing the overall and matrix backstress vectors are assumed identical to those of the corresponding stress vectors. They are listed here as follows:

$$\{\bar{\beta}^{\mathrm{m}}\} = [\bar{B}^{\mathrm{m}}]\{\bar{\beta}\} \tag{4a}$$

$$\{\mathrm{d}\bar{\beta}^{\mathrm{m}}\} = [\bar{B}^{\mathrm{mp}}]\{\mathrm{d}\bar{\beta}\}. \tag{4b}$$

The deviatoric backstress vectors are also given by:

$$\{\bar{\alpha}\} = [a]\{\bar{\beta}\} \tag{4c}$$

$$\{\bar{\alpha}^{\mathrm{m}}\} = [a]\{\bar{\beta}^{\mathrm{m}}\}. \tag{4d}$$

Substituting eqn (2a) into (3b), one obtains the following expression for the matrix deviatoric effective stress vector:

$$\{\bar{\tau}^{\mathrm{m}}\} = [a][\bar{B}^{\mathrm{m}}]\{\bar{\sigma}\}. \tag{5a}$$

Similarly, substituting eqn (4d) into (4a), one obtains the following expression for the matrix deviatoric effective backstress vector:

$$\{\bar{\alpha}^{\mathrm{m}}\} = [a][\bar{B}^{\mathrm{m}}]\{\bar{\beta}\}. \tag{5b}$$

The effective yield function $\bar{f}^{\mathrm{m}}$ for the matrix material is given here as a von Mises type with kinematic hardening

$$\bar{f}^{\mathrm{m}} = \tfrac{3}{2}\{\bar{\tau}^{\mathrm{m}} - \bar{\alpha}^{\mathrm{m}}\}^{\mathrm{T}}\{\bar{\tau}^{\mathrm{m}} - \bar{\alpha}^{\mathrm{m}}\} - \bar{\sigma}_0^{\mathrm{m}^2} \equiv 0, \tag{6}$$

where $\bar{\sigma}_0^{\mathrm{m}^2}$ is the yield strength of the matrix material. Substituting eqns (5a,b) into (6) and simplifying, one obtains the following expression for the effective yield function $\bar{f}$ for the overall composite system:

$$\bar{f} = \tfrac{3}{2}\{\bar{\sigma} - \bar{\beta}\}^{\mathrm{T}}[\bar{B}^{\mathrm{m}}]^{\mathrm{T}}[a][\bar{B}^{\mathrm{m}}]\{\bar{\sigma} - \bar{\beta}\} - \bar{\sigma}_0^{\mathrm{m}^2} \equiv 0. \tag{7}$$

Using the yield function expressions in eqns (6) and (7) and simplifying, one obtains the following formulae for the yield function derivative vectors $\{\partial\bar{f}^{\mathrm{m}}/\partial\bar{\sigma}^{\mathrm{m}}\}$ and $\{\partial\bar{f}/\partial\bar{\sigma}\}$:

$$\left\{\frac{\partial\bar{f}^{\mathrm{m}}}{\partial\bar{\sigma}^{\mathrm{m}}}\right\} = 3[a]\{\bar{\sigma}^{\mathrm{m}} - \bar{\beta}^{\mathrm{m}}\} \tag{8a}$$

$$\left\{\frac{\partial\bar{f}}{\partial\bar{\sigma}}\right\} = 3[\bar{B}^{\mathrm{m}}]^{\mathrm{T}}[a][\bar{B}^{\mathrm{m}}]\{\bar{\sigma} - \bar{\beta}\}. \tag{8b}$$

In fact, one can show [using only eqn (2a)] that the two derivative vectors given above are related by the following relation which is independent of the yield function:

$$\left\{\frac{\partial \bar{f}}{\partial \bar{\sigma}}\right\} = [\bar{B}^{\mathrm{m}}]^{\mathrm{T}} \left\{\frac{\partial \bar{f}^{\mathrm{m}}}{\partial \bar{\sigma}^{\mathrm{m}}}\right\}. \tag{9}$$

Effective strains and the flow rule

One now introduces the effective strain vectors $\{\bar{\varepsilon}^{\mathrm{m}}\}$ and $\{\bar{\varepsilon}^{\mathrm{f}}\}$ for the matrix and fibers, respectively, as follows:

$$\{\bar{\varepsilon}^{\mathrm{m}}\} = [\bar{\varepsilon}_1^{\mathrm{m}}\bar{\varepsilon}_2^{\mathrm{m}}\bar{\varepsilon}_3^{\mathrm{m}}]^{\mathrm{T}} \tag{10a}$$

$$\{\bar{\varepsilon}^{\mathrm{f}}\} = [\bar{\varepsilon}_1^{\mathrm{f}}\bar{\varepsilon}_2^{\mathrm{f}}\bar{\varepsilon}_3^{\mathrm{f}}]^{\mathrm{T}}. \tag{10b}$$

Similarly, the effective strain vector $\{\bar{\varepsilon}\}$ for the overall composite is given by:

$$\{\bar{\varepsilon}\} = [\bar{\varepsilon}_1\bar{\varepsilon}_2\bar{\varepsilon}_3]^{\mathrm{T}}. \tag{10c}$$

The elastic constant strain concentration matrices $[\bar{A}^{\mathrm{m}}]$ and $\{\bar{A}^{\mathrm{f}}]$ for the matrix and fibers, respectively, are defined by

$$\{\bar{\varepsilon}^{\mathrm{m}}\}' = [\bar{A}^{\mathrm{m}}]\{\bar{\varepsilon}\}' \tag{11a}$$

$$\{\bar{\varepsilon}^{\mathrm{f}}\} = [\bar{A}^{\mathrm{f}}]\{\bar{\varepsilon}\}, \tag{11b}$$

where the prime ′ indicates elastic strains. For the case of plastic loading or elastic unloading, the elasto-plastic strain concentration matrix $[\bar{A}^{\mathrm{mp}}]$ is defined by the following incremental relation:

$$\{\mathrm{d}\bar{\varepsilon}^{\mathrm{m}}\}'' = [\bar{A}^{\mathrm{mp}}]\{\mathrm{d}\bar{\varepsilon}\}'' \tag{11c}$$

where the double prime ″ indicates plastic strains. It is noticed that in eqn (11b), the total effective fiber strain vector $\{\bar{\varepsilon}^{\mathrm{f}}\}$ is used because the fibers undergo only elastic deformation. The elastic and plastic parts of the effective strain vectors are given by the additive decomposition

$$\{\mathrm{d}\bar{\varepsilon}\} = \{\mathrm{d}\bar{\varepsilon}\}' + \{\mathrm{d}\bar{\varepsilon}\}'' \tag{12a}$$

$$\{\mathrm{d}\bar{\varepsilon}^{\mathrm{m}}\} = \{\mathrm{d}\bar{\varepsilon}^{\mathrm{m}}\}' + \{\mathrm{d}\bar{\varepsilon}^{\mathrm{m}}\}''. \tag{12b}$$

An effective associated flow rule is used for the "undamaged" matrix material as follows:

$$\{\mathrm{d}\bar{\varepsilon}^{\mathrm{m}}\}'' = \mathrm{d}\bar{\lambda}^{\mathrm{m}} \left\{\frac{\partial \bar{f}^{\mathrm{m}}}{\partial \bar{\sigma}^{\mathrm{m}}}\right\} \tag{13}$$

where $\mathrm{d}\bar{\lambda}^{\mathrm{m}}$ is a scalar multiplier. Substituting eqns (11c) and (9) into (13) and simplifying, one obtains:

$$\{\mathrm{d}\bar{\varepsilon}\}'' = [\mathrm{d}\bar{\lambda}] \left\{\frac{\partial \bar{f}}{\partial \bar{\sigma}}\right\}, \tag{14}$$

where the multiplier matrix $[\mathrm{d}\bar{\lambda}]$ is given by:

$$[\mathrm{d}\bar{\lambda}] = \mathrm{d}\bar{\lambda}^{m}[\bar{A}^{mp}]^{-1}[\bar{B}^{m}]^{-T}. \tag{15}$$

Equation (14) clearly indicates a non-associated flow rule due to the presence of the 3×3 matrix $[\mathrm{d}\bar{\lambda}]$.

In the formulation, the "undamaged" matrix material undergoes kinematic hardening of the Prager–Ziegler type. This is represented by the evolution equation for the effective matrix backstress vector $\{\bar{\alpha}^{m}\}$ as follows:

$$\{\mathrm{d}\bar{\alpha}^{m}\} = \mathrm{d}\bar{\mu}^{m}\{\bar{\tau}^{m} - \bar{\alpha}^{m}\}, \tag{16}$$

where $\mathrm{d}\bar{\mu}^{m}$ is a scalar multiplier. In order to obtain a relation between the two scalar multipliers $\mathrm{d}\bar{\mu}^{m}$ and $\mathrm{d}\bar{\lambda}^{m}$, one equates the projection of the effective matrix incremental backstress vector on the yield surface $\bar{f}^{m}$ to $b\{\mathrm{d}\bar{\varepsilon}^{m}\}''$:

$$b\{\mathrm{d}\bar{\varepsilon}^{m}\}'' = \frac{\{\mathrm{d}\bar{\alpha}^{m}\}\left\{\dfrac{\partial\bar{f}^{m}}{\partial\bar{\sigma}^{m}}\right\}}{\left\{\dfrac{\partial\bar{f}^{m}}{\partial\bar{\sigma}^{m}}\right\}^{T}\left\{\dfrac{\partial\bar{f}^{m}}{\partial\bar{\sigma}^{m}}\right\}}\left\{\dfrac{\partial\bar{f}^{m}}{\partial\bar{\sigma}^{m}}\right\}, \tag{17}$$

where b is a constant material parameter determined from experiments. Post-multiplying eqn (17) by $\{\partial\bar{f}^{m}/\partial\bar{\sigma}^{m}\}^{T}$, simplifying and using eqn (8a), one obtains the desired relation as follows:

$$\mathrm{d}\bar{\mu}^{m} = 3b\,\mathrm{d}\bar{\lambda}^{m}. \tag{18}$$

It is noted that eqn (18) is valid only for the von Mises yield function $\bar{f}^{m}$ since it is used in the derivation. Next, one obtains an expression for the scalar multiplier $\mathrm{d}\bar{\lambda}^{m}$ using the consistency condition $\mathrm{d}\bar{f}^{m} = 0$ as follows:

$$\left\{\frac{\partial\bar{f}^{m}}{\partial\bar{\sigma}^{m}}\right\}^{T}\{\mathrm{d}\bar{\sigma}^{m}\} + \left\{\frac{\partial\bar{f}^{m}}{\partial\bar{\alpha}^{m}}\right\}^{T}\{\mathrm{d}\bar{\alpha}^{m}\} = 0. \tag{19}$$

Effective constitutive relation

Using the elastic relation ($[\bar{E}^{m}]$ is the effective elasticity matrix for the matrix material)

$$\{\mathrm{d}\bar{\sigma}^{m}\} = [\bar{E}^{m}]\{\mathrm{d}\bar{\varepsilon}^{m}\}' \tag{20a}$$

and substituting for $\{\mathrm{d}\bar{\varepsilon}^{m}\}'$ from eqn (12b), and for $\{\mathrm{d}\bar{\varepsilon}^{m}\}''$ from eqn (13), one obtains:

$$\{\mathrm{d}\bar{\sigma}^{m}\} = [\bar{E}^{m}]\left(\{\mathrm{d}\bar{\varepsilon}^{m}\} - \mathrm{d}\bar{\lambda}^{m}\left\{\frac{\partial\bar{f}^{m}}{\partial\bar{\sigma}^{m}}\right\}\right). \tag{20b}$$

Substituting eqns (16), (18) and (20b) into (19) and simplifying, one obtains the following expression for $\mathrm{d}\bar{\lambda}^{m}$:

$$\mathrm{d}\bar{\lambda}^{m} = \frac{1}{\bar{Q}^{m}}\left\{\frac{\partial\bar{f}^{m}}{\partial\bar{\sigma}^{m}}\right\}^{T}[\bar{E}^{m}]\{\mathrm{d}\bar{\varepsilon}^{m}\}, \tag{21a}$$

where the scalar quantity $\bar{Q}^{m}$ is given by

$$\bar{Q}^{\mathrm{m}} = 9\{\bar{\tau}^{\mathrm{m}} - \bar{\alpha}^{\mathrm{m}}\}^{\mathrm{T}}([\bar{E}^{\mathrm{m}}] + b[\mathbf{I}])\{\bar{\tau}^{\mathrm{m}} - \bar{\alpha}^{\mathrm{m}}\}. \tag{21b}$$

The expression of $\bar{Q}^{\mathrm{m}}$ given in eqn (21b) is valid only when using the von Mises yield function $\bar{f}^{\mathrm{m}}$. Next, one derives the effective matrix elasto-plastic stiffness matrix $[\bar{D}^{\mathrm{m}}]$. This is performed by substituting eqn (21a) into (20b) and simplifying. Therefore, one obtains

$$\{\mathrm{d}\bar{\sigma}^{\mathrm{m}}\} = [\bar{D}^{\mathrm{m}}]\{\mathrm{d}\bar{\varepsilon}^{\mathrm{m}}\}, \tag{22a}$$

where $[\bar{D}^{\mathrm{m}}]$ is given by

$$[\bar{D}^{\mathrm{m}}] = [\bar{E}^{\mathrm{m}}] - \frac{3}{\bar{Q}^{\mathrm{m}}}[\bar{E}^{\mathrm{m}}]\left\{\frac{\partial \bar{f}^{\mathrm{m}}}{\partial \bar{\sigma}^{\mathrm{m}}}\right\}\{\bar{\sigma} - \beta\}^{\mathrm{T}}[\bar{B}^{\mathrm{m}}]^{\mathrm{T}}[a][\bar{E}^{\mathrm{m}}]. \tag{22b}$$

The above equation can be used with any yield function $\bar{f}^{\mathrm{m}}$, except when using the specific expression of $\bar{Q}^{\mathrm{m}}$ given in eqn (21b).

Next, one derives an expression for the evolution of the effective overall backstress vector $\{\bar{\beta}\}$ based on eqn (16). Subtracting eqn (4a) from eqn (2a) and rewriting the resulting equation in incremental form, one obtains:

$$\{\mathrm{d}\bar{\sigma}^{\mathrm{m}} - \mathrm{d}\bar{\beta}^{\mathrm{m}}\} = [\bar{B}^{\mathrm{m}}]\{\mathrm{d}\bar{\sigma} - \mathrm{d}\bar{\beta}\}. \tag{23}$$

Upon plastic loading, one substitutes eqn (2c) into (23) and solves for $\{\mathrm{d}\bar{\beta}\}$ to obtain:

$$\{\mathrm{d}\bar{\beta}\} = ([\mathbf{I}] - [\bar{B}^{\mathrm{m}}]^{-1}[\bar{B}^{\mathrm{mp}}])\{\mathrm{d}\bar{\sigma}\} - [\bar{B}^{\mathrm{m}}]^{-1}\{\mathrm{d}\bar{\beta}\}^{\mathrm{m}}. \tag{24}$$

To find an expression for $\{\mathrm{d}\bar{\beta}^{\mathrm{m}}\}$ based on the Prager–Ziegler evolution law of eqn (16), one substitutes eqns (3b) and (4d) into (16) and simplifies. The resulting equation is:

$$\{\mathrm{d}\bar{\beta}^{\mathrm{m}}\} = \mathrm{d}\bar{\mu}^{\mathrm{m}}[\bar{B}^{\mathrm{m}}]\{\bar{\sigma} - \bar{\beta}\}. \tag{25}$$

Finally, substituting eqn (25) into (24), one obtains the following evolution law for the effective overall backstress vector $\{\bar{\beta}\}$:

$$\{\mathrm{d}\bar{\beta}\} = ([I] - [\bar{B}^{\mathrm{m}}]^{-1}[\bar{B}^{\mathrm{mp}}])\{\mathrm{d}\bar{\sigma}\} - \mathrm{d}\bar{\mu}^{\mathrm{m}}\{\bar{\sigma} - \bar{\beta}\}. \tag{26}$$

It is clear from eqn (26) that kinematic hardening of the composite material consists of two types. The first type is due to the kinematic hardening of the matrix material and is represented by the second term on the right-hand-side of eqn (26). The second type is represented by the first term on the right-hand-side of eqn (26) and is due to the interaction of the matrix and fibers. Therefore, the composite material will still undergo kinematic hardening (of the second type) even if the matrix material does not.

Equation (20a) introduced the effective elastic constitutive relation for the matrix. Similarly, one can introduce an effective elastic constitutive relation for the fibers in the form

$$\{\mathrm{d}\bar{\sigma}^{\mathrm{f}}\} = [\bar{E}^{\mathrm{f}}]\{\mathrm{d}\bar{\varepsilon}^{\mathrm{f}}\}, \tag{27}$$

where $[\bar{E}^{\mathrm{f}}]$ is the effective elasticity matrix for the fiber material and $\{\mathrm{d}\bar{\varepsilon}\}^{\mathrm{f}}$ consists entirely of elastic strain. The effective overall elastic constitutive relation can now be written in the form

$$\{\mathrm{d}\bar{\sigma}\} = [\bar{E}]\{\mathrm{d}\bar{\varepsilon}\}', \tag{28}$$

where $[\bar{E}]$ is the effective overall elasticity matrix. The matrix $[\bar{E}]$ is obtained from the matrices $[\bar{E}^{\mathrm{m}}]$ and $[\bar{E}^{\mathrm{f}}]$ as is shown shortly. Introducing the following relation between the effective overall and local stresses (Dvorak and Bahei-El-Din, 1979, 1982, 1987)

$$\{\mathrm{d}\bar{\sigma}\} = \bar{c}^{\mathrm{m}}\{\mathrm{d}\bar{\sigma}^{\mathrm{m}}\} + \bar{c}^{\mathrm{f}}\{\mathrm{d}\bar{\sigma}^{\mathrm{f}}\}, \tag{29}$$

where $\bar{c}^{\mathrm{m}}$ and $\bar{c}^{\mathrm{f}}$ are the matrix and fiber volume fractions, respectively ($\bar{c}^{\mathrm{m}} + \bar{c}^{\mathrm{f}} = 1$), one substitutes eqns (20a), (27) and (28) into (29) and simplifies to obtain:

$$[\bar{E}] = \bar{c}^{\mathrm{m}}[\bar{E}^{\mathrm{m}}][\bar{A}^{\mathrm{m}}] + \bar{c}^{\mathrm{f}}[\bar{E}^{\mathrm{f}}][\bar{A}^{\mathrm{f}}]. \tag{30}$$

In order to derive the effective overall elasto-plastic constitutive relation, one needs first to find an expression for the multiplier matrix $[\mathrm{d}\bar{\lambda}]$ of eqn (14) in terms of effective overall quantities. Therefore, one first invokes the consistency condition $\mathrm{d}\bar{f} = 0$.

$$\left\{\frac{\partial \bar{f}}{\partial \bar{\sigma}}\right\}^{\mathrm{T}} \{\mathrm{d}\bar{\sigma}\} + \left\{\frac{\partial \bar{f}}{\partial \bar{\beta}}\right\}^{\mathrm{T}} \{\mathrm{d}\bar{\beta}\} = 0. \tag{31}$$

Substituting for $\{\mathrm{d}\bar{\beta}\}$ from eqn (26), for $\{\mathrm{d}\bar{\sigma}\}$ from eqn (28), for $\{\mathrm{d}\bar{\varepsilon}\}'$ from eqn (12a), for $\{\mathrm{d}\bar{\varepsilon}\}''$ from eqn (14), and for $[\mathrm{d}\bar{\lambda}]$ from eqn (15), one obtains after simplifying and solving for $\mathrm{d}\bar{\lambda}^{\mathrm{m}}$:

$$\mathrm{d}\bar{\lambda}^{\mathrm{m}} = \{\bar{T}\}^{\mathrm{T}}[\bar{E}]\{\mathrm{d}\bar{\varepsilon}\}, \tag{32}$$

where the 3×1 vector $\{\bar{T}\}$ is given by:

$$\{\bar{T}\} = \frac{\left\{\dfrac{\partial \bar{f}}{\partial \bar{\sigma}}\right\} + ([I] - ([\bar{B}^{\mathrm{m}}]^{-1}[\bar{B}^{\mathrm{mp}}])^{\mathrm{T}})\left\{\dfrac{\partial \bar{f}}{\partial \bar{\beta}}\right\}}{\left\{\dfrac{\partial \bar{f}}{\partial \bar{\sigma}}\right\}^{\mathrm{T}} + \left\{\dfrac{\partial \bar{f}}{\partial \bar{\beta}}\right\}^{\mathrm{T}}([I] - [\bar{B}^{\mathrm{m}}]^{-1}[\bar{B}^{\mathrm{mp}}])[\bar{E}][\bar{A}^{\mathrm{mp}}]^{-1}[\bar{B}^{\mathrm{m}}]^{-\mathrm{T}}\left\{\dfrac{\partial \bar{f}}{\partial \bar{\sigma}}\right\} + 3b\left\{\dfrac{\partial \bar{f}}{\partial \bar{\beta}}\right\}^{\mathrm{T}}\{\bar{\sigma} - \bar{\beta}\}}. \tag{33}$$

One now starts with eqn (28) and substitutes for $\{\mathrm{d}\bar{\varepsilon}\}'$ from eqn (12a), for $\{\mathrm{d}\bar{\varepsilon}\}''$ from eqn (14), for $[\mathrm{d}\bar{\lambda}]$ from eqn (15) and for $\mathrm{d}\bar{\lambda}^{\mathrm{m}}$ from eqn (32) to obtain:

$$\{\mathrm{d}\bar{\sigma}\} = [\bar{D}]\{\mathrm{d}\bar{\varepsilon}\}, \tag{34a}$$

where the effective overall elasto-plastic stiffness matrix $[\bar{D}]$ is given by:

$$[\bar{D}] = [\bar{E}] - [\bar{E}][\bar{A}^{\mathrm{mp}}]^{-1}[\bar{B}^{\mathrm{m}}]^{-\mathrm{T}}\left\{\frac{\partial \bar{f}}{\partial \bar{\sigma}}\right\}\{\bar{T}\}^{\mathrm{T}}[\bar{E}]. \tag{34b}$$

Equation (34a) represents the effective elasto-plastic constitutive relation for the overall composite material.

Stresses in the damaged composite system

The second step of the formulation involves the incorporation of damage in the constitutive equations. This is performed by using the effective overall constitutive relation given in eqn (34a) and transforming it into a constitituve equation for the whole composite system. Therefore, all the quantities appearing in equations (34a,b) need to be transformed using the damage variable.

One first starts by using the linear transformation $[M]$ between the effective stress vector $\{\bar{\sigma}\}$ and the stress vector $\{\sigma\}$ as follows:

$$\{\bar{\sigma}\} = [M]\{\sigma\}, \tag{35}$$

where $[M]$ is a 3×3 matrix of the damage variables ϕ, ϕ_2 and ϕ_3 The matrix $[M]$ is represented in principal form as follows:

$$[M] = \begin{bmatrix} \frac{1}{1-\phi_1} & 0 & 0 \\ 0 & \frac{1}{1-\phi_2} & 0 \\ 0 & 0 & \frac{1}{1-\phi_3} \end{bmatrix} \tag{36}$$

and the stress vector $\{\sigma\} = [\sigma_1\ \sigma_2\ \sigma_3]^T$.

It is clear from eqn (36) that the matrix $[M]$ reduces to the identify matrix $[I]$ when there is no damage in the material, i.e. when $\phi_1 = \phi_2 = \phi_3 = 0$. On the other hand, the components of the matrix $[M]$ become very large when the material approaches complete rupture, i.e. when the values of ϕ_1, ϕ_2 and ϕ_3 approach 1. Actually, the values of ϕ_1, ϕ_2 and ϕ_3 do not need to approach 1 separately for rupture to occur. A representative scalar parameter (e.g. $\phi_{cr} = \sqrt{\phi_1^2+\phi_2^2+\phi_3^2}$) could be defined to characterize rupture. In the following formulation, the derivative matrix $d[M]$ is needed and is calculated using the chain rule as follows:

$$d[M] = \left[\frac{\partial M}{\partial \phi_1}\right]d\phi_1 + \left[\frac{\partial M}{\partial \phi_2}\right]d\phi_2 + \left[\frac{\partial M}{\partial \phi_3}\right]d\phi_3. \tag{37}$$

Substituting eqn (36) into (37) and simplifying, one obtains

$$d[M] = \begin{bmatrix} \frac{d\phi_1}{(1-\phi_1)^2} & 0 & 0 \\ 0 & \frac{d\phi_2}{(1-\phi_2)^2} & 0 \\ 0 & 0 & \frac{d\phi_3}{(1-\phi_3)^2} \end{bmatrix}. \tag{38}$$

Taking the derivative of eqn (35) and utilizing eqns (36) and (38) and simplifying, one obtains:

$$\{d\bar{\sigma}\} = \begin{Bmatrix} \frac{d\sigma_1}{1-\phi_1} + \frac{\sigma_1\, d\phi_1}{(1-\phi_1)^2} \\ \frac{d\sigma_2}{1-\phi_2} + \frac{\sigma_2\, d\phi_2}{(1-\phi_2)^2} \\ \frac{d\sigma_3}{1-\phi_3} + \frac{\sigma_3\, d\phi_3}{(1-\phi_3)^2} \end{Bmatrix}. \tag{39}$$

Using eqn (38), one can obtain the following expression for $d[M]\cdot\{\sigma\}$ which is used extensively in the derivatives that follow:

$$d[M]\cdot\{\sigma\} = [C^\sigma]\{d\phi\}, \tag{40a}$$

where the matrix $[C^\sigma]$ is given by:

$$[C^{\sigma}] = \begin{bmatrix} \frac{\sigma_1}{(1-\phi_1)^2} & 0 & 0 \\ 0 & \frac{\sigma_2}{(1-\phi_2)^2} & 0 \\ 0 & 0 & \frac{\phi_3}{(1-\phi_3)^2} \end{bmatrix} \tag{40b}$$

and the damage vector $\{d\phi\} = [d\phi_1\ d\phi_2\ d\phi_3]^T$. Similarly, one can derive the following equation for $d[M]\cdot\{\beta\}$:

$$d[M]\cdot\{\beta\} = [C^{\beta}]\{d\phi\}, \tag{41a}$$

where the matrix $[C^{\beta}]$ is given by:

$$[C^{\beta}] = \begin{bmatrix} \frac{\beta_1}{(1-\phi_1)^2} & 0 & 0 \\ 0 & \frac{\beta_2}{(1-\phi_2)^2} & 0 \\ 0 & 0 & \frac{\beta_3}{(1-\phi_3)^2} \end{bmatrix}. \tag{41b}$$

The expressions in eqs (40a) and (41a) are used extensively in the derivations below. However, the reader must keep in mind that these expressions are valid only when using principal values and the representation of $[M]$ given in eqn (36). In fact, these expressions cannot be easily generalized.

Substituting eqn (35) in (3a) and simplifying, one obtains the following relation for the effective overall deviatoric stress $\{\bar{\tau}\}$

$$\{\bar{\tau}\} = [N]\{\sigma\}, \tag{42a}$$

where the 3×3 matrix $[N]$ is given by

$$[N] = [a][M] \tag{42b}$$

and $[a]$ is the 3×3 constant matrix given in eqn (3c). The effective overall backstress vector $\{\beta\}$ is assumed to transform in a similar way to $\{\sigma\}$. Therefore, the following damage transformation equation is used [see eqn (35)]:

$$\{\bar{\beta}\} = [M]\{\beta\}. \tag{43a}$$

Substituting eqn (43a) into (4c) and simplifying, one obtains the following equation which is analogous to eqn (42a)

$$\{\bar{\alpha}\} = [N]\{\beta\}. \tag{43b}$$

Equations (42a) and (43b) represent the damage transformation equations for the effective overall stress and backstress vectors, respectively. They are used in the transformation of the yield function, the flow rule, the kinematic hardening rule and the constitutive relations. Starting with the effective yield function $\bar{f}$ given in eqn (7) and substituting for $\{\bar{\sigma}\}$ from eqn (35) and for $\{\bar{\beta}\}$ from eqn (43a) and simplifying, one obtains

$$f = \tfrac{3}{2}\{\sigma-\beta\}^{\mathrm{T}}[H]\{\sigma-\beta\}-\bar{\sigma}_0^{m^2} \equiv 0, \tag{44a}$$

where the 3×3 matrix $[H]$ is given by

$$[H] = [M]^{\mathrm{T}}[\bar{B}^{\mathrm{m}}]^{\mathrm{T}}[a][\bar{B}^{\mathrm{m}}][M]. \tag{44b}$$

Equation (44a) represents the yield function for the damaged composite system. The partial derivative $\{\partial f/\partial\sigma\}$ is now readily obtained from eqn (44a) as follows:

$$\left\{\frac{\partial f}{\partial \sigma}\right\} = 3[H]\{\sigma-\beta\}. \tag{45}$$

Using eqn (35), one can show, using the chain rule, that the following relation exists between the partial derivatives $\{\partial \bar{f}/\partial\bar{\sigma}\}$ and $\{\partial f/\partial\sigma\}$:

$$\left\{\frac{\partial f}{\partial \sigma}\right\} = [M]^{\mathrm{T}}\left\{\frac{\partial \bar{f}}{\partial \bar{\sigma}}\right\}. \tag{46}$$

The above relation is independent of the yield function.

Damage evolution

Several criteria are available in the literature for the description of damage evolution. The one chosen here is that proposed by Lee *et al.* (1985) for its simplicity and ease of integration in the proposed model. However, it should be emphasized that the model is so general that any viable damage criterion can be used.

Let g be the damage function given by Lee *et al.* (1985)

$$g = \tfrac{1}{2}\{\bar{\sigma}\}^{\mathrm{T}}[J]\{\bar{\sigma}\}-l_0^2-L(l) \equiv 0, \tag{47a}$$

where l is a scalar "overall" damage variable, and $[J]$ is a constant 3×3 matrix given by

$$[J] = \begin{bmatrix} 1 & \mu & \mu \\ \mu & 1 & \mu \\ \mu & \mu & 1 \end{bmatrix} \tag{47b}$$

and μ is a constant damage parameter, $-0.5 \leqslant \mu \leqslant 1.0$. The matrix representation of $[J]$ given in equation (47b) applies only for the problem considered here. A more general representation can be found in the recent work of Lee *et al.* (1985). Substituting eqn (35) into (47a) and simplifying, one obtains:

$$g = \tfrac{1}{2}\{\sigma\}^{\mathrm{T}}[M]^{\mathrm{T}}[J][M]\{\sigma\}-l_0^2-L(l) \equiv 0. \tag{48a}$$

Using equation (48a), one can readily determine the following partial derivatives of g:

$$\frac{\partial g}{\partial L} = -1 \tag{48b}$$

$$\left\{\frac{\partial g}{\partial \sigma}\right\} = [M]^{\mathrm{T}}[J][M]\{\sigma\}. \tag{48c}$$

In order to determine the evolution equation for the damage vector $\{\phi\}$, one starts with the power of dissipation Π given by

$$\Pi = \{\sigma\}^{\mathrm{T}}\{\mathrm{d}\varepsilon\}'' + \{\sigma\}^{\mathrm{T}}\{\mathrm{d}\phi\} - L\,\mathrm{d}l. \tag{49}$$

The problem of damage evolution now reduces to the problem of extremization of Π subject to the constraints $f = 0$ and $g = 0$. One, therefore, introduces the objective function ψ given by

$$\psi = \Pi - \mathrm{d}\lambda_1 f - \mathrm{d}\lambda_2 g. \tag{50}$$

where $\mathrm{d}\lambda_1$ and $\mathrm{d}\lambda_2$ are scalar Lagrange multipliers. Using the conditions $\{\partial\psi/\partial\sigma\} = \{0\}$ and $\partial\psi/\partial L = 0$ and simplifying, one obtains

$$\{\mathrm{d}\phi\} = \mathrm{d}\lambda_2\left\{\frac{\partial g}{\partial\sigma}\right\} \tag{51a}$$

$$\mathrm{d}l = -\mathrm{d}\lambda_2\frac{\partial g}{\partial L}. \tag{51b}$$

Substitution of eqn (48b) into (51b) results in $\mathrm{d}\lambda_2 = \mathrm{d}l$. Substituting this result into eqn (51a), one obtains

$$\{\mathrm{d}\phi\} = \mathrm{d}l\left\{\frac{\partial g}{\partial\sigma}\right\}. \tag{52}$$

In order to obtain the scalar damage multiplier $\mathrm{d}l$, one needs to apply the consistency condition $\mathrm{d}g = 0$.

$$\left\{\frac{\partial g}{\partial\sigma}\right\}^{\mathrm{T}}\{\mathrm{d}\sigma\} + \left\{\frac{\partial g}{\partial\phi}\right\}^{\mathrm{T}}\{\mathrm{d}\phi\} + \frac{\partial g}{\partial l}\mathrm{d}l = 0. \tag{53}$$

Substituting for $\{\mathrm{d}\phi\}$ from eqn (52), using $\partial g/\partial l = (\partial g/\partial L)\,(\mathrm{d}L/\mathrm{d}l) = -\mathrm{d}L/\mathrm{d}l$, and solving for $\mathrm{d}l$, one obtains:

$$\mathrm{d}l = \frac{1}{r}\left\{\frac{\partial g}{\partial\sigma}\right\}^{\mathrm{T}}\{\mathrm{d}\sigma\}. \tag{54a}$$

where the scalar quantity r is given by:

$$r = \frac{\mathrm{d}L}{\mathrm{d}l} - \left\{\frac{\partial g}{\partial\phi}\right\}^{\mathrm{T}}\left\{\frac{\partial g}{\partial\sigma}\right\}. \tag{54b}$$

Finally, substituting eqn (54a) into (52), one obtains the required evolution equation for the damage vector $\{\phi\}$:

$$\{\mathrm{d}\phi\} = \left(\frac{1}{r}\left\{\frac{\partial g}{\partial\sigma}\right\}^{\mathrm{T}}\left\{\frac{\partial g}{\partial\sigma}\right\}\right)\{\mathrm{d}\sigma\}. \tag{55}$$

Equation (55) can be rewritten simply as $\{\mathrm{d}\phi\} = r^*\{\mathrm{d}\sigma\}$, where r^* is the scalar quantity shown in parentheses in eqn (55). It should be noted that eqn (55) represents a set of three simultaneous ordinary differential equations in the variables ϕ_1, ϕ_2 and ϕ_3. This set of differential equations will be used later in the numerical solution of the problem.

Elastic constitutive relation in the damaged composite system

The next step is the derivation of the elastic constitutive relation. This is accomplished by first obtaining the damage transformation equation for the elastic strain rate vector $\{d\varepsilon\}'$. Starting with the effective elastic energy $\bar{U}$ given by

$$\bar{U} = \tfrac{1}{2}\{\bar{\sigma}\}^{T}\{\bar{\varepsilon}\}' \tag{56}$$

and using the hypothesis of elastic energy equivalence ($\bar{U} = U$), one obtains:

$$\{\bar{\sigma}\}^{T}\{\bar{\varepsilon}\}' = \{\sigma\}^{T}\{\varepsilon\}'. \tag{57}$$

Substituting eqn (35) into (57) and solving for $\{\bar{\varepsilon}\}'$, one obtains:

$$\{\bar{\varepsilon}\}' = [M]^{-T}\{\varepsilon\}'. \tag{58}$$

Using the method used in the derivation of eqns (40a) and (41a), one can show that:

$$d[M]^{-T}\{\varepsilon\}' = [C^{\varepsilon}]\{d\phi\}, \tag{59a}$$

where $[C^{\varepsilon}]$ is obtained from eqn (40b) by replacing σ with ε'. Taking the derivative of eqn (58) and substituting eqn (59a) in the resulting expression, one obtains the damage transformation eqn for $\{d\varepsilon\}'$:

$$\{d\bar{\varepsilon}\}' = [M]^{-T}\{\varepsilon\}' + [C^{\varepsilon}]\{d\phi\}. \tag{59b}$$

In order to find a relation between $\{d\bar{\sigma}\}$ and $\{d\sigma\}$, one takes the derivative of eqn (35) and substitutes eqns (40a) and (55) into the resulting expression. After simplification, one obtains:

$$\{d\bar{\sigma}\} = [M^*]\{d\sigma\}, \tag{60a}$$

where the 3×3 matrix $[M^*]$ is given by

$$[M^*] = [M] + \frac{1}{r}[C^{\sigma}]\left(\left\{\frac{\partial g}{\partial \sigma}\right\}^{T}\left\{\frac{\partial g}{\partial \sigma}\right\}\right). \tag{60b}$$

Finally, substituting eqns (55), (59b) and (60a) into the effective elastic constitutive relation given in eqn (28) and simplifying, one obtains the elastic constitutive relation for the damaged composite system as follows:

$$\{d\sigma\} = [E]\{d\varepsilon\}', \tag{61a}$$

where the 3×3 damage-elasticity matrix is given by

$$[E] = \left([M^*] - \frac{1}{r}\left(\left\{\frac{\partial g}{\partial \sigma}\right\}^{T}\left\{\frac{\partial g}{\partial \sigma}\right\}\right)[\bar{E}][C^{\varepsilon}]\right)^{-1}[\bar{E}][M]^{-T}. \tag{61b}$$

Elasto-plastic constitutive relation in the damaged composite system

The kinematic hardening rule given in eqn (26) can now be transformed to the damaged composite system. Substituting eqns (35) and (43a) into eqn (26), simplifying and solving for $\{d\beta\}$, one obtains:

$$\{d\beta\} = ([M]^{-1} - [M]^{-1}[\bar{B}^{m}]^{-1}[\bar{B}^{mp}])[M]\{d\sigma\}$$
$$-(([M]^{-1} - [M]^{-1}[\bar{B}^{m}]^{-1}[\bar{B}^{mp}])[dM] + d\bar{\mu}^{m}[I])\{\sigma\}$$
$$-([M]^{-1}[dM] - d\bar{\mu}^{m}[I])\{\beta\}. \tag{62}$$

The additive decomposition of the strain rate tensor is taken in the form

$$\{d\varepsilon\} = \{d\varepsilon\}' + \{d\varepsilon\}''. \tag{63}$$

It can be shown that the above decomposition is compatible with the decomposition in eqn (12a).

The flow rule for the damaged composite system is taken in the form

$$\{d\varepsilon\}'' = [d\lambda]\left\{\frac{\partial f}{\partial \sigma}\right\}. \tag{64}$$

The above equation clearly provides for a non-associated flow rule. This is in agreement with the recent results of Stolz (1986) where it was shown that an associated flow rule may not be derivable for damaged materials. The multiplier matrix $[d\lambda]$ is determined from the consistency condition $df = 0$ as follows:

$$\left\{\frac{\partial f}{\partial \sigma}\right\}^{T}\{d\sigma\} + \left\{\frac{\partial f}{\partial \beta}\right\}^{T}\{d\beta\} = 0. \tag{65}$$

Substituting for $\{d\beta\}$ from eqn (62), for $\{d\sigma\}$ from eqn (61a), for $\{d\varepsilon\}'$ from eqn (63), for $\{d\varepsilon\}''$ from eqn (64), for $\{d\phi\}$ from eqn (55), and using eqns (40a) and (41a), and simplifying, one obtains:

$$\{\gamma\}^{T}\{d\sigma\} = d\bar{\mu}^{m}\left\{\frac{\partial f}{\partial \beta}\right\}^{T}\{\sigma - \beta\}, \tag{66a}$$

where the 3×1 vector $\{\gamma\}$ is given by:

$$\{\gamma\} = \left\{\frac{\partial f}{\partial \sigma}\right\} + [M]^{T}([I] - [\bar{B}^{m}]^{-1}[\bar{B}^{mp}])^{T}[M]^{-T}\left\{\frac{\partial f}{\partial \beta}\right\}$$
$$-\frac{1}{r}\left(\left\{\frac{\partial g}{\partial \sigma}\right\}^{T}\left\{\frac{\partial g}{\partial \sigma}\right\}\right)([C^{\sigma}]^{T}([I] - [\bar{B}^{m}]^{-1}[\bar{B}^{mp}])^{T} + [C^{\beta}]^{T})[M]^{-T}\left\{\frac{\partial f}{\partial \beta}\right\}. \tag{66b}$$

The solution of eqn (66a) for $d\bar{\mu}^{m}$ yields:

$$d\bar{\mu}^{m} = \frac{\{\gamma\}^{T}\{d\sigma\}}{\left\{\frac{\partial f}{\partial \beta}\right\}^{T}\{\sigma - \beta\}}. \tag{67}$$

Substituting eqn (67) into (18), solving for $d\bar{\lambda}^{m}$ and substituting the result into eqn (15), one obtains the following expression for the multiplier matrix $[d\bar{\lambda}]$:

$$[\mathrm{d}\bar{\lambda}] = \frac{\{\gamma\}^{\mathrm{T}}[E]\{\mathrm{d}\varepsilon\}}{3b\left\{\dfrac{\partial f}{\partial \beta}\right\}^{\mathrm{T}}\{\sigma-\beta\}}[\bar{A}^{\mathrm{mp}}]^{-1}[\bar{B}^{\mathrm{m}}]^{-\mathrm{T}} - \frac{\{\gamma\}^{\mathrm{T}}[E]\{\mathrm{d}\varepsilon\}''}{3b\left\{\dfrac{\partial f}{\partial \beta}\right\}^{\mathrm{T}}\{\sigma-\beta\}}[\bar{A}^{\mathrm{mp}}]^{-1}[\bar{B}^{\mathrm{m}}]^{-\mathrm{T}}. \tag{68}$$

Equating the plastic energy of dissipation $(1/2)\{\sigma\}^{\mathrm{T}}\{\mathrm{d}\varepsilon\}''$ in the damaged configuration with the plastic energy of dissipation $(1/2)\{\bar{\sigma}\}^{\mathrm{T}}\{\mathrm{d}\bar{\varepsilon}\}''$ in the effective configuration, and using eqn (35), one obtains:

$$\{\mathrm{d}\bar{\varepsilon}\}'' = [M]^{-\mathrm{T}}\{\mathrm{d}\varepsilon\}''. \tag{69}$$

Equation (69) is the damage transformation equation for the plastic strain rate vector. Finally, in order to derive the elasto-plastic constitutive relation for the damaged composite system, one substitutes eqns (60a), (12a), (59b), (69), (55), (61a) (for $\{\mathrm{d}\varepsilon\}'$) into (34a) and simplifies to obtain:

$$\{\mathrm{d}\sigma\} = [D]\{\mathrm{d}\varepsilon\}, \tag{70a}$$

where the damage-elasto-plastic stiffness matrix is given by:

$$[D] = \left([M] + \frac{1}{r}\left(\left\{\frac{\partial g}{\partial \sigma}\right\}^{\mathrm{T}}\left\{\frac{\partial g}{\partial \sigma}\right\}\right)[C^{\sigma}]\right)^{-1}[\bar{D}][M]^{-\mathrm{T}}. \tag{70b}$$

Finally, one needs to rewrite eqn (62) in a form suitable for numerical implementation. In order to rewrite it in the required incremental form, one substitutes eqns (40a) and (41a) into (62) to obtain

$$\{\mathrm{d}\beta\} = [X^*]\{\mathrm{d}\sigma\} + [Y^*]\{\mathrm{d}\phi\}, \tag{71a}$$

where the 3×3 matrices $[X^*]$ and $[Y^*]$ are given by:

$$[X^*] = [I] - [M]^{-1}[\bar{B}^{\mathrm{m}}]^{-1}[\bar{B}^{\mathrm{mp}}][M] - \frac{\left\{\dfrac{\partial f}{\partial \sigma}\right\}^{\mathrm{T}}[\bar{B}^{\mathrm{m}}]^{-1}[\bar{A}^{\mathrm{mp}}]^{-\mathrm{T}}[E]^{\mathrm{T}}\{\gamma\}}{3b\left\{\dfrac{\partial f}{\partial \beta}\right\}^{\mathrm{T}}\{\sigma-\beta\}}\{\sigma-\beta\}\{\gamma\}^{\mathrm{T}} \tag{71b}$$

$$[Y^*] = -[M]^{-1}([C^{\sigma}] - [\bar{B}^{\mathrm{m}}]^{-1}[\bar{B}^{\mathrm{mp}}][C^{\sigma}] + [C^{\beta}]). \tag{71c}$$

It is noticed that eqn (71a) represents a set of three simultaneous differential equations in β_1, β_2, and β_3. This set will be used in the next section for the numerical solution of the problem.

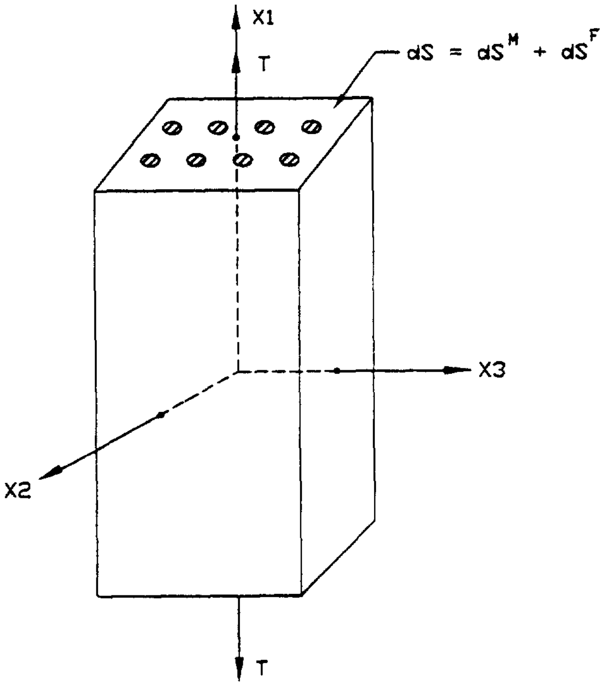

Fig. 1. Unidirectional thin lamina under uniaxial tension.

NUMERICAL IMPLEMENTATION

Consider a unidirectional fiber-reinforced thin composite lamina that is subjected to a uniaxial tensile force T along the x_1-direction as shown in Fig. 1. The matrix is assumed to be elasto-plastic and the fibers elastic and cylindrical in shape. The fibers are also assumed to be continuous, perfectly aligned and symmetrically distributed along the x_1-axis. For this problem, the stress vector $\{\sigma\}$ is given by

$$\{\sigma\} = [\sigma\, 0\, 0]^{T}, \tag{72}$$

where σ is the uniaxial stress obtained by dividing T by the cross-sectional area of the lamina. Substituting $\{\sigma\}$ into eqn (70a), one writes the constitutive equation for this problem as follows:

$$\begin{Bmatrix} d\varepsilon_1 \\ d\varepsilon_2 \\ d\varepsilon_3 \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix}^{-1} \begin{Bmatrix} d\sigma \\ 0 \\ 0 \end{Bmatrix}, \tag{73}$$

where $d\sigma$ is the increment of uniaxial stress. Since we have only one independent component, σ, in the stress vector, a system of simultaneous ordinary differential equations can be written for this problem with σ as the independent variable. In this way, this problem can be solved numerically using a suitable differential equation solver without the use of finite elements. Let the matrix $[S]$ denote the inverse of $[D]$, and rewrite eqn (73) as follows:

$$\begin{Bmatrix} d\varepsilon_1 \\ d\varepsilon_2 \\ d\varepsilon_3 \end{Bmatrix} = \begin{Bmatrix} S_{11} \\ S_{21} \\ S_{31} \end{Bmatrix} d\sigma. \tag{74}$$

Equation (74) represents the first set of the governing system of differential equations for this problem. It should be mentioned that the expressions of S_{11}, S_{21} and S_{31} are obtained by numerically inverting the elasto-plastic matrix $[D]$. The matrix $[D]$ is obtained using eqn (70b) with the condition that the stress vector is given by eqn (72).

The second set of differential equations is obtained for the evolution of the backstress vector $\{\beta\}$ given by eqn (71a). However, eqn (71a) must be rewritten in the required format to be used in the system of differential equations. In other words, the right-hand-side should be a function of the independent variable $\mathrm{d}\sigma$. Therefore, the second term of the right-hand-side of eqn (71a) will be rewritten in terms of the vector $\{\mathrm{d}\sigma\}$. Substituting for $\{\mathrm{d}\phi\}$ from eqn (55) into (71a) and simplifying, one obtains

$$\{\mathrm{d}\beta\} = [Z^*]\{\mathrm{d}\sigma\}, \tag{75a}$$

where the matrix $[Z^*]$ is given by:

$$[Z^*] = [X^*] + \frac{1}{r}\left(\left\{\frac{\partial g}{\partial \sigma}\right\}^{\mathrm{T}}\left\{\frac{\partial g}{\partial \sigma}\right\}\right)[Y^*]. \tag{75b}$$

Substituting eqn (72) into (75a), one rewrites the resulting equation as follows:

$$\begin{Bmatrix} \mathrm{d}\beta_1 \\ \mathrm{d}\beta_2 \\ \mathrm{d}\beta_3 \end{Bmatrix} = \begin{Bmatrix} Z^*_{11} \\ Z^*_{21} \\ Z^*_{31} \end{Bmatrix} \mathrm{d}\sigma. \tag{76}$$

Equation (76) represents the second set of differential equations required for the solution of this problem. Finally, the last set of differential equations uses the evolution of the damage vector $\{\phi\}$ as given by eqn (55). Equation (55) is rewritten in the required format as follows:

$$\begin{Bmatrix} \mathrm{d}\phi_1 \\ \mathrm{d}\phi_2 \\ \mathrm{d}\phi_3 \end{Bmatrix} = \frac{1}{r}\left(\left\{\frac{\partial g}{\partial \sigma}\right\}^{\mathrm{T}}\left\{\frac{\partial g}{\partial \sigma}\right\}\right)\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}\mathrm{d}\sigma. \tag{77}$$

The three sets of equations: (74), (76) and (77) represent the governing system of differential equations for this problem. Taking the independent variable σ as the time t, the governing system of differential equations is given by

$$\mathrm{d}\varepsilon_1/\mathrm{d}t = S_{11} \tag{78a}$$

$$\mathrm{d}\varepsilon_2/\mathrm{d}t = S_{21} \tag{78b}$$

$$\mathrm{d}\varepsilon_3/\mathrm{d}t = S_{31} \tag{78c}$$

$$\mathrm{d}\beta_1/\mathrm{d}t = Z^*_{11} \tag{78d}$$

$$\mathrm{d}\beta_2/\mathrm{d}t = Z^*_{21} \tag{78e}$$

$$\mathrm{d}\beta_3/\mathrm{d}t = Z^*_{31} \tag{78f}$$

$$\mathrm{d}\phi_1/\mathrm{d}t = \frac{1}{r}\left\{\frac{\partial g}{\partial \sigma}\right\}^{\mathrm{T}}\left\{\frac{\partial g}{\partial \sigma}\right\} \tag{78g}$$

$$\mathrm{d}\phi_2/\mathrm{d}t = 0 \tag{78h}$$

$$\mathrm{d}\phi_3/\mathrm{d}t = 0. \tag{78i}$$

Equations (78) form a system of nine simultaneous ordinary differential equations that can be solved numerically using a Runge–Kutta type method. In the numerical solution, it is assumed that the elastic strains are infinitesimal; therefore, they are neglected. Consequently, the solution scheme starts at the initiation of yielding. This means that the initial conditions for this problem are zero strains, backstresses, and damage variables. Therefore, in this problem, damage is initiated at the same time yielding starts; though this may not be the case in a general problem where the amount of elastic strain may be significant. As initial conditions to the boundary value problem, all nine dependent variables $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \beta_1, \beta_2, \beta_3, \phi_1, \phi_2, \phi_3)$ are taken to be zero while the initial value of the independent variable $\sigma = \sigma_0 \neq 0$. The initial value σ_0 is needed for the solution of the differential equations. It is computed by setting the yield function f of eqn (44a) to be equal to zero. It should also be pointed out that at the start of plasticity and damage, the backstresses and damage variables are also zero. Substituting zero for $\{\beta\}$ in eqn (44a), one obtains the following condition at yielding:

$$\tfrac{3}{2}\{\sigma\}^{\mathrm{T}}[H]\{\sigma\} - \bar{\sigma}_0^{\mathrm{m}^2} = 0, \tag{79}$$

where $[H]$ is given by eqn (44b). However, since $\phi_1 = \phi_2 = \phi_3 = 0$ at yielding, the matrix $[M]$ becomes the identity matrix $[I]$. Substituting $[I]$ for $[M]$ in eqn (44b), one obtains:

$$[H] = [\bar{B}^{\mathrm{m}}]^{\mathrm{T}}[a][\bar{B}^{\mathrm{m}}]. \tag{80}$$

The stress concentration matrix $[\bar{B}^{\mathrm{m}}]$ is obtained using the Voigt and Mori–Tanaka models as discussed shortly while $[a]$ is the constant matrix given in eqn (3c).

Substituting $[\sigma_0\, 0\, 0]^{\mathrm{T}}$ for $\{\sigma\}$ in eqn (79) and solving for σ_0, one obtains:

$$\sigma_0 = \sqrt{\frac{2}{3H_{11}}}\,\bar{\sigma}_y^{\mathrm{m}}, \tag{81}$$

where H_{11} is the first term in the matrix $[H]$ of eqn (80), and $\bar{\sigma}_y^{\mathrm{m}}$ is the yield strength of the matrix material. Equation (81) represents the initial condition for the uniaxial stress σ to be used in the solution of the differential equations.

Equations (78) are solved simultaneously using the IMSL routine DIVPRK. This routine uses a Runge–Kutta–Verner fifth-order and sixth-order method for the solution of the differential equations. Figure 2 shows a schematic diagram of the numerical computations. In the determination of the stress and strain concentration matrices, two different models are used. The first is the Voigt model which is based on the assumption that the strains in the matrix, fibers and composite are equal. The second model is the Mori–Tanaka model which uses the Eshelby tensor and the theory of inclusions and inhomogeneities. The Mori–Tanaka model is more sophisticated than the Voigt model but the latter is considered here for comparison. Details about the two models can be found in the papers of Dvorak and Bahei-El-Din (1979, 1982, 1987), Voyiadjis and Kattan (1992, 1993a), and Mori and Tanaka (1973). Details about the numerical scheme used in calculating the Eshelby tensor are found in the papers of Gavazzi and Lagoudas (1990), and Lagoudas *et al.* (1991).

The lamina consists of matrix and fibers with volume fractions 55 and 45%, respectively. The material properties used are: $\bar{E}^{\mathrm{m}} = 84.1$ GPa, $\bar{v}^{\mathrm{m}} = 0.3$, $\bar{E}^{\mathrm{f}} = 414$ GPa, $\bar{v}^{\mathrm{f}} = 0.22$, $\bar{E}_{11} = 200$ GPa, $\bar{E}_{22} = 137$ GPa, $\bar{v}_{12} = 0.27$, $\bar{v}_{23} = 0.31$, and $\bar{G}_{12} = 52.6$ GPa. The yield strength of the matrix material is 51 ksi (0.35 GPa). The damage parameters are $\mu = 0.5$ and $\partial L/\partial l = 1.0 \times 10^{12}$. Using eqn (81), one finds that the stress at which yielding occurs is $\sigma_0 = 1$ GPa for the Voigt model and $\sigma_0 = 0.4$ GPa for the Mori–Tanaka model. It is noted that the material yields at a higher yield stress when using the Voigt model because of the assumption of equal strains in the material thus making it stiffer. In the numerical calculations, the stress is increased monotonically starting from the yield stress σ_0 in increments of 1 GPa for a total of 100 increments. The tolerance factor for convergence of the iterative scheme is taken as 0.005. The results are shown in Figs 3–5.

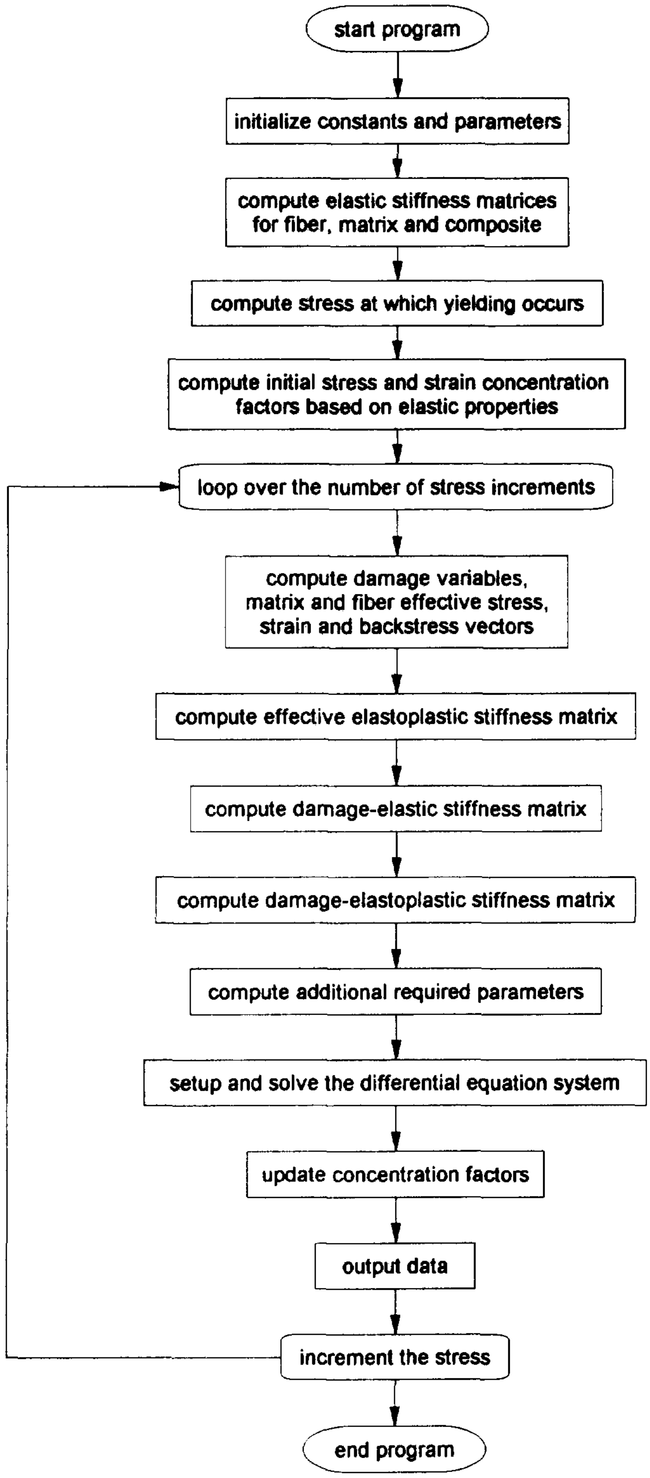

Fig. 2. Schematic diagram of the numerical computations.

In Fig. 3, the variation of the damage variable ϕ_1 is shown vs. the strain ε_1. It is clear that the value of ϕ_1 is monotonically increasing for both the Voigt and Mori–Tanaka model although the rate of increase of damage is higher when the Voigt model is used. This may be attributed to the use of constant concentration factors when the Voigt model is used. In the Mori–Tanaka model, the concentration matrices change depending on the stiffness of the material. The values of ϕ_2 and ϕ_3 are identically zero, therefore, no plots for these

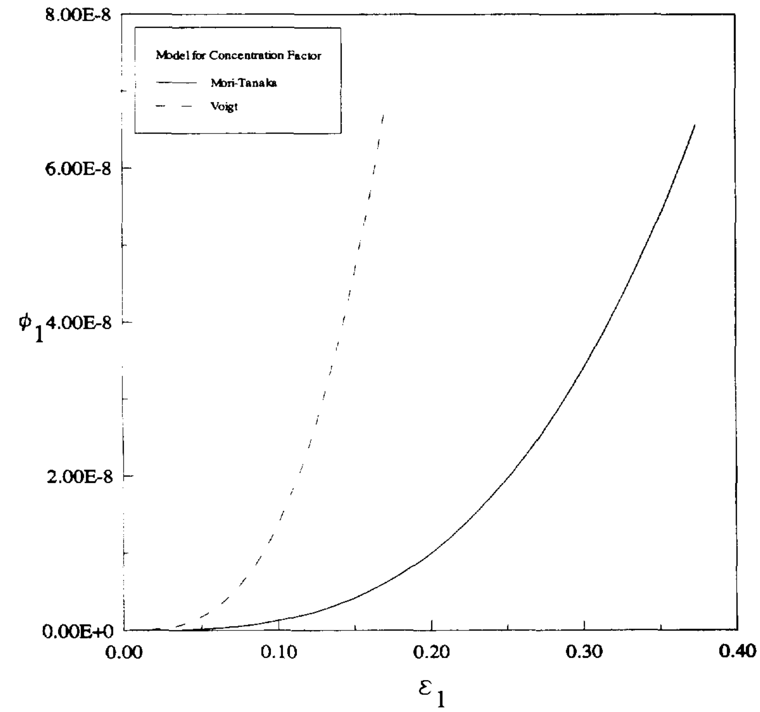

Fig. 3. Damage variable ϕ_1 vs. strain ε_1.

damage variables are shown. The variations of the backstresses β_1 and β_2 are shown vs. the strain ε_1 in Figs 4 and 5, respectively. It is noticed that the Voigt model gives higher values of the backstress β_1. However, the backstress β_2 vanishes when using the Voigt model. This is again attributed to the simplifying assumptions used for the Voigt model. It is apparent that the values of μ and $\partial L/\partial l$ provide for very small values of the damage variable ϕ_1. It should be emphasized that the solution of practical problems in this area requires the use

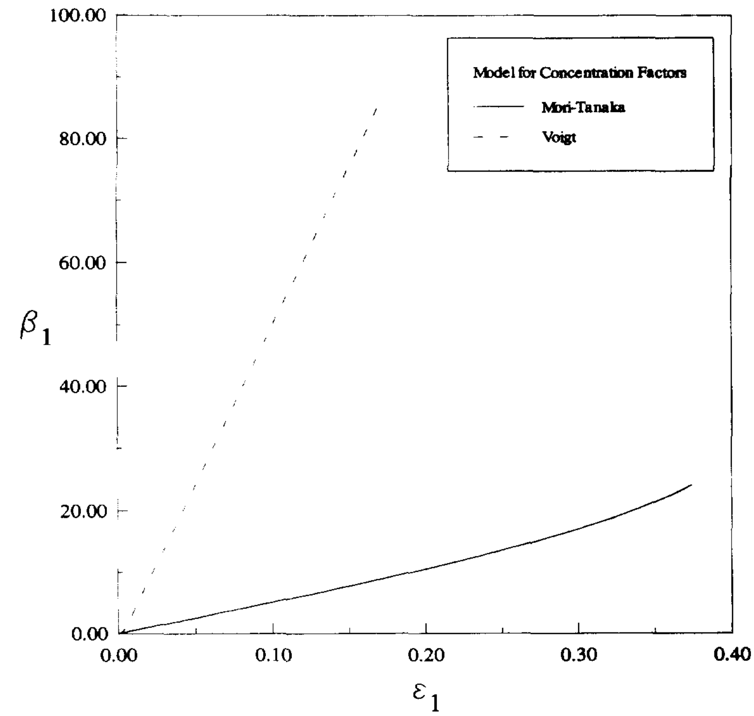

Fig. 4. Backstress β_1 vs. strain ε_1.

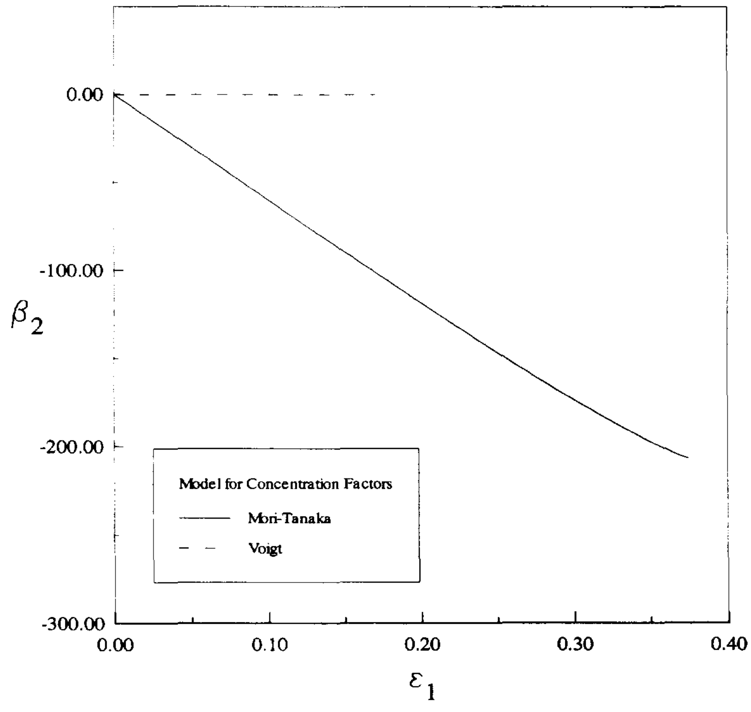

Fig. 5. Backstress β_2 vs. strain ε_1.

of the finite element method which may well be the subject of future research. Therefore, the authors choose to solve this simple problem of one lamina as an example in order to demonstrate how the model can be used without resorting to finite elements.

PHYSICAL CHARACTERIZATION AND EXPERIMENTAL DETERMINATION OF DAMAGE

In this section, a physical interpretation of the damage tensor ϕ is presented for the case of micro-cracks. Experimental investigations and procedures for the determination of damage are presented by Voyiadjis and Venson (1995) for the macro- and micro-analysis of a SiC–titanium aluminide metal matrix composite. Uniaxial tension tests were performed on laminate specimens of two different layups. Dogbone shaped flat specimens are fabricated from each of the layups. Specimens for the different layups are then loaded to various load levels ranging from rupture load down to 70% of the rupture load at room temperature. Through this experimental procedure, damage evolution is experimentally evaluated through a quantitative micro-analysis technique. The micro-analysis is performed using scanning electron microscopy on three mutually perpendicular representative cross-sections of all specimens for the qualitative and quantitative determination of damage.

A new damage tensor proposed by Voyiadjis and Venson (1995) is defined here for a general state of loading based upon experimental observations of crack densities on three mutually perpendicular cross-sections of the specimens. The damage tensor, ϕ, is defined as a second-rank tensor represented by the 3×3 matrix:

$$[\phi] = \begin{bmatrix} \bar{\rho}_x\bar{\rho}_x & \bar{\rho}_x\bar{\rho}_y & \bar{\rho}_x\bar{\rho}_z \\ \bar{\rho}_y\bar{\rho}_x & \bar{\rho}_y\bar{\rho}_y & \bar{\rho}_y\bar{\rho}_z \\ \bar{\rho}_z\bar{\rho}_x & \bar{\rho}_z\bar{\rho}_y & \bar{\rho}_z\bar{\rho}_z \end{bmatrix}, \tag{82}$$

where $\bar{\rho}_i = (i = x, y, z)$ is the normalized crack density on a cross-section whose normal is along the i-axis. For the uniaxial tension of one lamina, the diagonal form of the above matrix may be used. The crack density on the representative volume element for the ith cross-section is calculated as follows:

$$\bar{\rho}_i = \frac{\rho_i}{m\rho^*} \tag{83}$$

$$\rho_i = \frac{l_i}{A_i}. \tag{84}$$

where l_i is the total length of the cracks on the ith cross-section for each constituent, A_i is the ith cross-sectional area for the representative volume element, m is a normalization factor chosen so that the values of the damage variable ϕ fall within the expected range $0 \leqslant \phi_{ij} < 1$, and ρ^* is as defined below:

$$\rho^* = \sqrt{\rho_{x_{max}}^2 + \rho_{y_{max}}^2 + \rho_{z_{max}}^2}. \tag{85}$$

where ρ_i is the value of l_i/A_i at the maximum load (rupture). The damage tensor obtained experimentally from eqn (82) is then used in the constitutive equations to predict the mechanical behavior of the composite system.

The scanning electron microscope is used in order to quantify the damage tensor ϕ expressed by eqn (82). This is performed at various load levels ranging from rupture load down to 70% of the rupture load at room temperature. The damage tensor ϕ is determined experimentally by Voyiadjis and Venson (1995) for two types of laminate layups $(0/90)_s$ and $(\pm 45)_s$, each consisting of four plies. These layups are examined both numerically and experimentally (Voyiadjis and Venson, 1995).

These results are then used to calculate the values of the damage variable ϕ^r based on eqn (82). In this way, damage–strain curves are generated for each layup orientation. These damage values can then be used in the constitutive model to accurately predict the mechanical behavior of metal matrix composites. The final results are presented in a paper by Voyiadjis and Venson (1995). The reader is referred to this paper for a more complete discussion on the physical characterization of the damage tensor ϕ.

CONCLUSION

The overall approach to damage is investigated for a uniaxially loaded unidirectional thin lamina. The lamina is made of elastic continuous fibers and an elasto-plastic matrix. The formulation is presented for a general state of deformation and damage, except when using the derivative of the damage effect tensor, which is used in terms of its principal values. The governing differential equations for the damaged system are formulated and solved numerically using a Runge–Kutta–Verner fifth order and sixth order method. The results are compared using both the Voigt and Mori–Tanaka models for the calculation of the concentration factors. It is interesting to note how this problem can be solved numerically without the use of finite elements. The complete finite element implementation of the model will be presented in a subsequent paper.

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